A Method of Calculating the Characteristic Impedance of a Strip Transmission Line to a Given Degree of Accuracy*

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Summary—The calculation of the characteristic impedance of the strip transmission line TEM-mode can be reduced to the solution of a two-dimensional potential equation with the strip cross section determining the boundary conditions.

Usually this potential equation is solved by conformal mapping, but only the most simple shapes permit exact mapping. Approximations may require considerable work and their accuracy is uncertain.

This paper describes an alternative numerical method which is particularly suitable for boundaries consisting of any number of straight lines and right angles.

It is based on relaxation methods, but by using also variational principles it derives an approximate value for the impedance, and an upper and lower bound with a difference as small as desirable.

IN the last few years, strip transmission line has become increasingly popular for use in transmission lines, filters, mixers, and other components in the kilomegacycle range. For all these applications, values of the characteristic impedance Z of the strip are required, but are often difficult to obtain with good accuracy.

It is the purpose of this paper to describe a new method of numerical calculation which is accurate and simple to use for any strip cross section.

The strip, having a conductor insulated from ground, supports a TEM wave if the medium is homogeneous. The calculation of a TEM field reduces to that of a 2-dimensional Laplace equation with boundaries given by the strip cross section, and the propagation constant depends only upon the medium.[1-2]

Based on this, the characteristic impedance Z is expressed by:

$$Z = \sqrt{\frac{L}{C}} = \frac{\sqrt{\mu\epsilon}}{C} \tag{1}$$

$$C = -\epsilon \oint_H \frac{\partial \phi}{\partial n} ds, \tag{2}$$

$$\phi \cdots \text{solution of } \Delta\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

for the given boundary condition.

The integration extends along the "hot" conductor.

C may be identified with the capacity of the strip per unit length; but it might also be used to solve related problems in which the potential equation defines other physical quantities, like magnetic flux, fluid flow, or heat flux.

In order to obtain C, approximation methods, based on estimates for the "fringe" capacity were first introduced.[2,3] Later a number of workers used conformal mapping,[4-13] but only few cases, usually assuming infinitely thin conductors, can be calculated in closed form; even they lead to rather complicated expressions in elliptic functions. Otherwise approximations must be made to obtain usable results.

We note that the main effort of the conformal method is directed toward solving the Laplace equation point by point, or in other words, to describe the electromagnetic field in every point of the cross section. This field is usually of no interest by itself, but a close approximation of the field seems necessary to obtain a close value for C.

Variational methods[14] are known to be numerically useful for just such a type of problem and very recently

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they have been applied to certain TEM structures, as for instance trough lines.[15–18] They permit a good estimate even if a relatively crude trial function is substituted into a "stationary integral." However, variational methods have two disadvantages. Good trial functions must permit easy numerical work, they should contain a large number of parameters which can be chosen arbitrarily, and they should cover a fairly general case. Such functions are not always easily found. Also, the calculated approximation is actually an upper bound for the stationary integral, but little advantage is obtained from this fact as long as no lower bound is known.

In this paper it is intended to derive two stationary integrals, which give both an upper and a lower bound for the characteristic impedance Z. Both integrals permit relatively easy numerical evaluation when trial functions obtained by a process similar to relaxation are used.

We shall consider the region R of the complex plane which is bounded by curves G and H, corresponding to the cross sections of ground plate and "hot" conductor, and by the two sides of cuts F_i which are introduced with the purpose of making the region R of the complex plane simply connected. This is illustrated in Fig. 1.

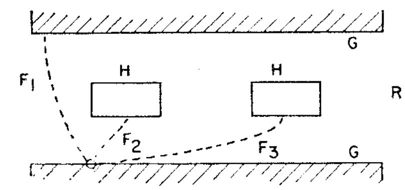

Fig. 1—Strip cross section, showing construction of the simply connected region R.

In this region R we shall define a potential function ϕ and a stream function ψ as follows:

The potential function ϕ is defined by:

$$\phi = 0 \text{ on } G \tag{3a}$$

$$\phi = 1 \text{ on } H \tag{3b}$$

$$\phi \text{ and } \frac{\partial\phi}{\partial n} \text{ are continuous across the cuts } F \tag{3c}$$

$$\Delta\phi = 0 \text{ in } R. \tag{4}$$

Because of (3a)–(3c) the expression $\phi(\partial\phi/\partial n)$ will be equal to the integrand on H, disappear on G, and cancel on both sides of F. Consequently, we may express C by

$$\frac{C}{\epsilon} = -\oint_H \frac{\partial\phi}{\partial n} ds = -\oint_{F,G,H} \phi \frac{\partial\phi}{\partial n} ds. \tag{5}$$

The subscript F, G, H, indicates that the integration extends along the whole contour of the simply connected region R. On (5) we may apply the Gauss Theorem, and using (4) we obtain

$$\frac{C}{\epsilon} = \iint_R (\nabla\phi)^2 dA \tag{6}$$

with the notation

$$(\nabla\phi)^2 = \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2$$

(because this is a two-dimensional problem) and:

$$dA = dxdy.$$

The stream function ψ is defined as the conjugate function to $(\epsilon/C)\phi$, or in other words by the Cauchy-Riemann equations:

$$\frac{\partial\phi}{\partial x} = \frac{C}{\epsilon}\frac{\partial\psi}{\partial y} \tag{7a}$$

$$\frac{\partial\phi}{\partial y} = -\frac{C}{\epsilon}\frac{\partial\psi}{\partial x} \tag{7b}$$

from which it follows immediately that ψ too is a solution of the Laplace equation:

$$\Delta\psi = 0 \tag{8}$$

and

$$(C/\epsilon)^2(\nabla\psi)^2 = (\nabla\phi)^2. \tag{9}$$

ψ is uniquely determined except for an additive constant, which may be chosen arbitrarily.

The boundary conditions for ψ follow from those for ϕ, (3a)–(3c)

$$\text{on } G \text{ and } H: \frac{\partial\psi}{\partial n} = 0 \tag{10a}$$

$$\text{on } F_i: \quad \frac{\partial\psi}{\partial n} \quad \text{continuous.} \tag{10b}$$

Across F_i the function ψ will be discontinuous. (The cuts F_i have been introduced for this purpose: otherwise ψ could not be uniquely defined.) However, the "jump" of ψ will have a constant value along the cut F_i. We shall denote this constant by writing F_i as index to the function ψ:

$$\text{along } F_i: \psi/_{12} = \psi_{F_i} = \text{constant.} \tag{10c}$$

We consider now only those cuts F_h which lead from G to H. For these cuts we shall calculate the expression $\sum\psi_{F_h}$ as follows: If we integrate along a closed curve around H:

$$\oint_H \frac{\partial\psi}{\partial s} ds$$

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[17] N. Tomcio, "The Characteristic Impedance of a Transmission Line Consisting of a Ribbon in a Rectangular Trough," University of Toronto, Toronto, Ont. Can., Dept. of Elec. Eng. Res. Rep.; 1954.

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this integral does not disappear, because of the discontinuities ψ_{F_h}. However,

$$\oint \frac{\partial \psi}{\partial s} ds + \sum \psi_{F_h} = 0. \quad (11)$$

Substituting from (7) and (2) we see

$$-\frac{C}{\epsilon}\frac{\partial \psi}{\partial s} = \frac{\partial \phi}{\partial n}$$

$$\sum \psi_{F_h} = -\oint \frac{\partial \psi}{\partial s} ds = -\frac{1}{-C/\epsilon}\oint_H \frac{\partial \phi}{\partial n} ds = 1. \quad (12)$$

(Provided that consistent sign conventions are used, *e.g.*, counterclock integration, crossing F_h from 1 to 2, normal direction pointing outside from conductor.) Usually there will be only one hot conductor, and one cut F, and the sum $\sum \psi_{F_h} = 1$ will have only one term. However, (13) will be derived to cover also the case of several conductors, which of course may be used to support a number of TEM modes. Some such structures have been recently described.[18–20]

Finally we express C/ϵ by ψ, substituting (9) into (6):

$$\frac{C}{\epsilon} = \frac{1}{\iint_R (\nabla \psi)^2 dA}. \quad (13)$$

While (13) as well as (6) seems to be unnecessarily more complicated than (2) we note as previously shown[14] that both area integrals are stationary. Consequently, even crude approximations of ϕ and ψ will give close upper bounds for the area integrals, and with it upper and lower bounds for Z. The approximations have to fulfill the same boundary conditions as ϕ and ψ and they must be continuous; they need not be solutions of the Laplace equations. We may formulate:

a) If a function U is continuous and if it takes the values $U=1$ along the hot conductor(s) and $U=0$ along ground, then U may be used as trial function to calculate an upper bound for C by

$$\frac{C}{\epsilon} \leqq \iint_R (\nabla U)^2 dA. \quad (14)$$

The equal sign holds only if $\Delta U = 0$.

b) If a function V is continuous in a simply connected region R, which is obtained by arbitrary cuts F_i from the part of the complex plane which is bounded by the conductor and ground plate cross section, and if the jump in V across any cut F_i is, for this particular cut, a constant V_{F_i}, and if the sum

$$\sum_h V_{F_h} = 1,$$

(the summation extending over all cuts leading from ground to a hot conductor), then V may be used as trial function to calculate a lower bound for C by

$$\frac{C}{\epsilon} \geqq \frac{1}{\iint_R (\nabla V)^2 dA}. \quad (15)$$

The equal sign holds only if $\Delta V = 0$.

Physically the integrals (6) and (13) are twice the electrostatic energy ($\frac{1}{2}CU^2$) for unit potential drop, and twice the electromagnetic energy ($\frac{1}{2}LI^2$) for unit current in the conductor(s); both are expressed by the energy density of the respective fields. The stationary character of the integrals may be explained by noting that any approximation of the field introduces additional space charges or currents which can only increase the total energy.

The lower bound is then obtained from L by eliminating Z from

$$Z = \frac{\sqrt{\epsilon\mu}}{C} = \sqrt{\frac{L}{C}}. \quad (1)$$

However, because of the importance of the relations (14) and (15) they will be derived separately in Appendix II; this will also give a better understanding of the restrictions in the choice of U and V.

The bounds obtained by the integrals are remarkably close if we construct suitable trial functions. This will be done now for U; the construction of V is completely analogous. As (14) and (15) can be used to check the accuracy of the result, we may feel free to choose the trial functions not so much for close approximation of the field but for numerical convenience:

a) We chose a Cartesian grid of square mesh with N meshes of side length h which is conveniently located with regard to boundaries and symmetry lines.

b) Within each square k of the grid, we define a different trial function $U_k(x, y)$ as the bilinear expression:

$$U_k(x, y) = B_{k1} \cdot xy + B_{k2} \cdot x + B_{k3} \cdot y + B_{k4}.$$

U_k is a potential function with four available parameters B_k, which shall be chosen in such a way that U_k interpolates linearly between the (arbitrarily chosen) values of U at the grid corners, which we shall name "grid values." Then the functions U_k will join continuously across the grid lines.

c) The function U, which is constructed to be equal to each U_k in its square, is then a continuous function and suitable as trial function.

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This trial function contains the arbitrarily chosen grid values as independent parameters which shall be chosen later so that a minimum condition will be fulfilled. Therefore, it is logical to express by these grid values the contribution of the kth square to the stationary integral. Substituting gives at first an expression in B_k;

$$\iint_R (\nabla U)^2 dA = \sum_{k=1}^{n} \iint_{R_k} (\nabla U_k)^2 dx dy$$

$$\iint_{R_k} (\nabla U_k)^2 dx dy$$

$$= \int_{-h/2}^{+h/2} \int_{-h/2}^{+h/2} [(B_{k1}y + B_{k2})^2 + (B_{k1}x + B_{k3})^2] dy dx$$

$$= h^2\left[\frac{h^2 B_{k1}^2}{6} + B_{k2}^2 + B_{k3}^2\right] \tag{16}$$

but we wanted to express the integrals in terms of the grid values U_{k1}, *etc.*

After further substituting and rearranging, the final equation may be written in two forms:

$$\iint (\nabla U)^2 dA = \sum \left[\tfrac{1}{6}(U_1 - U_2 + U_3 - U_4)^2 + \tfrac{1}{2}(U_1 - U_3)^2 + \tfrac{1}{2}(U_2 - U_4)^2\right] \tag{17}$$

$$= \sum \tfrac{1}{6}[(U_1 - U_2)^2 + (U_2 - U_3)^2 + (U_3 - U_4)^2 + (U_4 - U_1)^2 + 2(U_1 - U_3)^2 + 2(U_2 - U_4)^2]. \tag{18}$$

Eq. (17) is useful for numerical evaluation as shown in Appendix I. However, (18) is helpful in understanding the additional restrictions we can impose on the grid values so that the integral becomes a minimum.

It is of course desirable to use such grid values that the integral (14) has the lowest possible value. As this integral in any case must be larger than C, the lowest integral gives also the lowest error. In (18) the terms in the brackets contain twice the squares of the changes of U along the diagonals, but only once the square of the change along the side of the mesh. However, the same amount again is contributed by the side of the next mesh square, as shown in Fig. 2.

Consequently each grid value U_{k1} contributes with equal weight the squares of the eight differences between U_{k1} and its eight neighboring grid values, to the quadratic terms of the sum (18). The minimum of these 8 terms is obtained when the grid value U_{k1} is the arithmetic mean of its 8 surrounding grid values. This gives one linear equation. Similar equations are desirable for all other grid values. This results in a system of linear equations which has only the one solution corresponding to the lowest value which the upper bound for C can take if any function defined for the given grid is used as trial function.

If the number of grid values is large, these equations may be solved by the relaxation methods introduced and described in detail by Southwell and his coworkers[21–23] except that conventional relaxation is based on the average of only four neighboring values and is of little known accuracy. Also, because our integrals cancel the first-order error, the relaxation must be carried out only to one or two digits accuracy for our case, so that there is less numerical work than in a conventional relaxation.

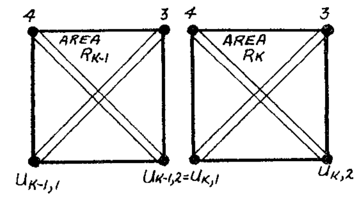

Fig. 2—Each value U_{k1} is the average of eight surrounding values.

Conclusion

In conclusion we may describe the numerical method as follows:

1) Make a low accuracy relaxation plot for unity potential drop.
2) Repeat the plot, but for unity potential discontinuity.
3) Square the differences $(U_1 - U_2 + U_3 - U_4)$, etc., and summate for both plots according to (17). The normalized impedance must then lie between the inverse of the sum of the first plot and the sum of the second plot.

Appendix I

Example

Suppose we have the symmetrical strip of the cross section, Fig. 3, and we want its impedance for the odd mode. Because of conventional symmetry considerations and the availability of a solution for one strip (due to Bates[4]) shown in Fig. 4, we have to solve numerically only for twice the piece of cross section which is shown enlarged in Fig. 5, covered by a grid of ten meshes and bounded by equipotential and symmetry lines. The field and the characteristic impedance for the cross section, as shown in Figs. 5–7, will be evaluated numerically. This is intended as an example, illustrating the

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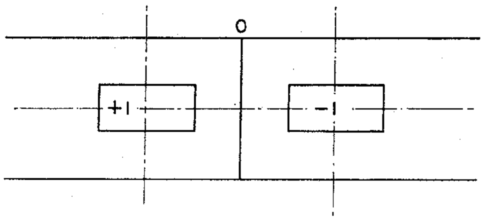

Fig. 3—Double conductor strip.

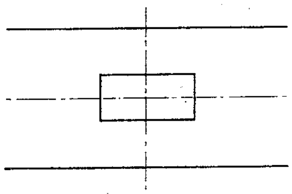

Fig. 4—Strip solved by Bates.[4]

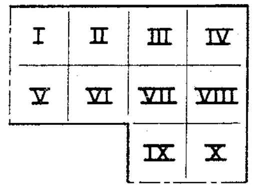

Fig. 5—Detail of Fig. 3, showing numbered mesh.

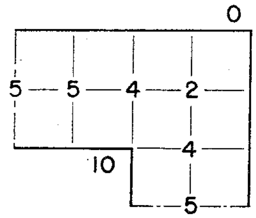

Fig. 6—Relaxation solution for the "grid values" of Fig. 5. Each grid value is approximately the average of the eight neighboring grid values.

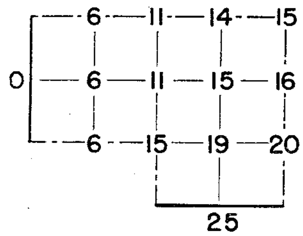

Fig. 7—Relaxation solution for the grid values of Fig. 5 after interchanging equipotential and symmetry lines.

numerical work required. Fig. 6 shows a very rough approximation of the potential at the corners of the grid, which we call "grid values." Each "grid value" is obtained from the condition that it must be approximately the average of its eight neighboring grid values; approximately means accurate within one or two digits. With some practice the grid values may be just written down without further calculation.

Fig. 7 shows a similar plot, except that equipotential and symmetry lines are interchanged. Decimal points are omitted in both plots. Their proper location is given by the requirement that Fig. 6 shows a field with unity potential drop, and Fig. 7, a field with unity sum of potential discontinuities. For each mesh, we calculate the following expressions in its 4 grid values U:

$$(U_1 - U_2 + U_3 - U_4)^2, \quad (U_1 - U_3)^2, \quad (U_2 - U_4)^2$$

and add these terms for all N mesh squares, according to (17) as tabulated and evaluated in Table I.

As our approximation is still very crude, we consider it to fit a similar plot evaluating the left half of Fig. 4. This plot need not be worked out because we obtain its results from the Bates solution[4] which, in this particular case, gives $Z_B = 51.5$ ohms. The (with 377 ohm) normalized values for the impedance and admittance of the left half may be added to the values of our plot as if we had obtained them by continuing our plot into the Bates region.

The results of the numerical calculation are two values, one larger than $377/Z$, the other larger than $Z/377$ (the normalized impedance of the strip). This can be rewritten as $Z = 48.6$ ohms with an accuracy just shown to be better than ± 2 per cent. A more refined plot with 40 meshes and 2 digit accuracy has been worked out and gave $Z = 48.5$ ohms ± 0.8 per cent. This illustrates that it is not difficult to get good results.

Appendix II

For proof of (14) write U as sum of the correct solution ϕ and an error ϕ',

$$U = \phi + \phi'.$$

The boundary conditions for the error ϕ' are:

$$\phi' = 0 \text{ on } G$$

$$\phi' = 0 \text{ on } H$$

$$\phi' \text{ and } \frac{\partial \phi'}{\partial n} \text{ continuous across } F.$$

We note that the last condition is not very stringent, as the location of the cuts F is arbitrary. Consequently,

$$\oint_{FGH} \phi' \frac{\partial \phi}{\partial n} ds = 0$$

and, using again Gauss' theorem and (4) this becomes

$$\iint \nabla\phi \nabla\phi' dA = 0$$

for any ϕ' which only fulfills the boundary conditions and is continuous and of limited variation so that the Gauss theorem can be used. Expanding the expression on the right side of (14), we get

$$\iint_R (\nabla U)^2 dA$$

$$= \iint_R (\nabla\phi)^2 dA + 2\iint_R \nabla\phi\nabla\phi' dA + \iint_R (\nabla\phi')^2 dA,$$

TABLE I

NUMERICAL EVALUATION OF Z

Mesh	$(U_1-U_2+U_3-U_4)^2$	$(U_1-U_3)^2$	$(U_2-U_4)^2$	$(V-V_2+V_3-V_4)^2$	$(V_1-V_3)^2$	$(V_2-V_4)^2$	Square terms of final equation
I	0	5	25	0	36	36	
II	1	25	16	0	25	25	
III	4	16	4	1	9	16	
IV	4	4	0	0	0	4	
V	0	25	25	0	36	36	
VI	1	36	25	16	25	81	
VII	16	64	0	0	0	64	
VIII	4	16	4	0	9	25	
IX	1	36	25	16	36	100	
X	1	25	16	1	25	36	
	3)32	272	140	3)34	201	423	Sum up:—
	11			11			$\frac{1}{3}\sum(\ )^2+\sum(\ )^2+\sum(\ )^2$
	272			201			
	140			423			
	423			635			
	4.23			0.0635 ←			—Restore decimal point
	3.66 ←						—Add known solution for remaining half of field
		377		0.0683←	Z		
	7.89 >	Z		0.1318 >	377		$\sqrt{\mu/\epsilon}=377\ \Omega$

The 2 inequalities give: $Z=48.6\ \Omega\pm2$ per cent
A more refined plot with half the mesh size and 2 digit accuracy gives: $Z=48.5\ \Omega\pm0.8$ per cent

and substituting (6) gives

$$\iint_R (\nabla U)^2 dA = \frac{C}{\epsilon} + \iint_R (\nabla\phi')^2 dA \tag{19}$$

from which (14) follows immediately. Regarding accuracy, (19) shows also how the error in C depends on the error of the approximation U of the potential function ϕ. Finally we may drop the requirements for limited variation; otherwise the integrals would not even be finite, much less a minimum.

For proof of (15) write V as sum of the correct solution ψ and an error ψ':

$$V = \psi + \psi'$$

$$\sum_n V_{F_h} = 1$$

$$\sum_n \psi_{F_h}' = 0 \tag{20}$$

in which V_{F_h} and ψ_{F_h}' denote the jump of V and ψ' across the cuts F_h (which are crossed when circling all H boundaries).

We now have to calculate

$$\oint_{FGH} \psi' \frac{\partial\psi}{\partial n} ds.$$

Because $\partial\psi/\partial n=0$ on G and H (10a) this becomes:

$$\oint_{FGH} \psi' \frac{\partial\psi}{\partial n} ds = \sum_i \int_{F_i} \psi_{F_i}' \frac{\partial\psi}{\partial n} ds$$

$$= \sum_i \left(\psi_{F_i}' \int_{F_i} \frac{\partial\psi}{\partial n} ds\right).$$

Substitute from (7)

$$\frac{C}{\epsilon}\int_{F_i} \frac{\partial\psi}{\partial n} ds = \int_{F_i} \frac{\partial\phi}{\partial s} ds.$$

This integral can have only two values:

= 0 if F_i connects two boundaries of equal potential, or

= 1 if F_i connects from a G boundary to an H boundary

Only the latter ones are crossed by the contour around all H conductors; therefore only these terms contribute to the sum, which we indicate by changing the index from i to h:

$$\oint_{FGH} \psi' \frac{\partial\psi}{\partial n} ds = \sum_i \psi_{F_i}' \int_{F_i} \frac{\partial\psi}{\partial n} ds$$

$$= \frac{1}{C/\epsilon}\sum_h \psi_{F_h}' \int_{F_h} \frac{\partial\phi}{\partial s} ds = \frac{1}{C/\epsilon}\sum_h \psi_{F_h}' = 0.$$

If ψ' is continuous and of limited variation we can apply the Gauss theorem and get

$$\iint_R \nabla\psi\nabla\psi' dA = 0$$

for any function ψ' which need only fulfill (20). From this follows

$$\iint_R (\nabla V)^2 dA = \frac{1}{C/\epsilon} + \iint_R (\nabla\psi')^2 dA$$

which gives us (15) immediately.

APPENDIX III

This Appendix deals with the convergence of the bounds when evaluated by relaxation methods.

For all practical purposes, it will be sufficient to calculate the upper and lower bounds, which give directly the total possible errors and, if necessary, to go to one finer plot. However, this can be repeated to any given accuracy; in other words, it is sufficient to refine the plot if the difference between the bound and the correct value is to be made smaller than any given value. This shall now be shown briefly.

The error in (19) consists of two parts: 1) Error due to incomplete relaxation, 2) Inherent error due to size of the grid.

The first error is essentially numerical and shall not be of concern here. It is assumed that a sufficiently accurate relaxation is available. For the second error, however, we will assume that we know the exact solution ϕ and we will take arbitrarily the values of the exact solution as grid values U_{k1} etc., and get

$$U_k = \phi_k.$$

With (19) this error becomes

$$\iint (\nabla\phi')^2 dA.$$

But on the grid corners $\phi_k' = 0$; and because ϕ is continuous and differentiable the integral, and with it the error for this arbitrarily chosen plot, will disappear in the limit if the grid is made sufficiently fine.

Of course the plot for which it was just proved that its error can be made negligible is not the plot obtained by relaxation. However, it is a pleasant feature of the variational method as compared to conventional relaxation that this does not really matter because the error due to the completely relaxed plot must be still smaller than the error from any other plot. This completes the proof.

Ferrite Line Width Measurements in a Cross-Guide Coupler*

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Summary—**Theoretical and experimental results are presented to show that the line width and the *g* factor of a spherical ferrite sample can be measured in a cross-guide coupler. The method is much easier to instrument than the usual cavity method and the measurements are much easier to perform. Experimental verification with a cavity perturbation system indicates that the measured quantities are sufficiently accurate for most purposes.**

Introduction

Recent work by the author with ferrite directional couplers indicated that such devices[1] might be useful in measuring the line width and the *g* factor of ferrites. Earlier unpublished work by the author in this field showed that such devices were not practical for measuring the components of the susceptibility tensor of a ferrite sphere because of the extreme sensitivity that was required in the detecting system. However, the measurement of the line width and the *g* factor depends upon relative power measurements. Thus, it was felt that the couplers offered considerable promise of yielding accurate data with a minimum of effort. A second method was also desirable in order to verify data on line widths and *g* factors obtained with a cavity-perturbation system. It was not intended that the new method be highly accurate, but rather that it be simple and capable of yielding reliable comparative data during the development of optimum manufacturing techniques.

The method chosen is extremely simple and uses a cross-guide directional coupler with a round, centered hole in the common broad wall. The wall thickness at the coupling aperture is about half the normal waveguide wall thickness and the hole is of such a diameter that the coupling is 40–50 db. For the X-band test coupler presently used, the hole diameter is $\frac{1}{8}$ inch and the wall thickness at the hole is 0.020 inch. The X-band test coupler is illustrated in Fig. 1. The two waveguides are soldered together and an access hole is provided for inserting the sample into the coupling hole. The fit of the cover plate on the access hole is not critical since any leakage of power through it is unimportant in relative power measurements of this type as long as the leakage power remains constant while the measurements are being taken. The ferrite sample is glued symmetrically in the coupling hole with Duco Cement, which had no noticeable effect on the measurements. However, the placement of the sample in the coupling hole had some effect on the level of the coupled power but no noticeable effect on the line width or the *g* factor.

The usual method for measuring the microwave susceptibility and the effective *g* factor of a ferrite depends upon the complex frequency perturbation of a resonant cavity. The method is quite popular but fairly difficult to instrument. It also has other disadvantages:

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[1] D. C. Stinson, "Ferrite directional couplers with off-center apertures," IRE Trans. on Microwave Theory and Techniques, vol. MTT-6, pp. 332–333; July, 1958.